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QUANTIZATION AMBIGUITY AND NON-TRIVIAL VACUUM STRUCTURE

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It is pointed out that there is an ambiguity in quantization of any system whose configuration space has a non-trivial topology characterized by a Chern number. In field theories this ambiguity manifests itself through the existence of θ sectors. The point of view adopted gives a simple interpretation of the difference between the temporal and Coulomb gauge descriptions of instantons. The general ideas are exemplified in the O(3) non-linear σ model in two dimensions.

There has been a considerable interest in the past few years in the non-trivial vacuum structure of non-Abelian gauge theories as signalized by the BPST instanton [1]. Historically this vacuum structure was first exposed in the temporal gauge where there is enough gauge degree of freedom to allow one to interpret the instantons as a link between topologically inequivalent classical vacua (n-vacua) [2]. Such a quasi-classical picture of vacuum tunneling, although heuristically very appealing, is clearly gauge dependent as is evident from working in the Coulomb gauge with strong boundary conditions where the classical vacuum is unique [3–5]. On the other hand the θ vacua, which in the temporal gauge are just coherent superpositions of the n-vacua [2], obviously are not just gauge artifacts. The purpose of this note is to characterize in a more intrinsic fashion the θ vacua as resulting from an ambiguity in quantization of classical systems whose configuration space has a non-trivial topology. From this point of view it becomes evident that the non-linear O(3) σ model in two dimensions also has a non-trivial vacuum structure resulting from the existence of configurations with non-vanishing Chern number.

The abovementioned quantization ambiguity can be exemplified in a simple quantum mechanical system, where the momentum can be realized in the coordi-

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nate representation as

$$p_i = -i \frac{\partial}{\partial q_i} + A_i(q) , \tag{1}$$

with

$$\partial_i A_j - \partial_j A_i = 0 .$$

If the configuration space of the q_i's is simply connected, then by means of a gauge transformation

$$\psi(q) \to e^{-i\varphi(q)} \psi(q) \equiv \psi'(q) ,$$

with φ and A_i related by $A_i = \partial_i \varphi$, one comes back to the usual representation of the momentum. On the other hand, if the space is not simply connected, then the description in which the momentum is given by $-i\partial/\partial q_i$ may correspond to wave functions which are no longer single valued. For example in the case of one degree of freedom with q an angular variable, if one choses $A = \theta/2\pi$, where θ is a constant, one has that $\psi'(q)$ satisfies a quasi-periodic boundary condition

$$\psi'(q + 2\pi) = e^{-i\theta} \psi'(q) . \tag{2}$$

Eq. (2) exhibits the basic mechanism giving rise to a non-trivial vacuum structure in field theories [6]. If we "unwind" the angular variable q by extending its range of definition to the whole real line, it is already possible in this example to introduce metastable n-vacua whose coherent superposition will form the true ground state of the theory. Such a procedure corresponds to an enlargement of configuration space, allowing for a quasi-classical understanding of the vacuum structure, paralleling the temporal gauge description in gauge theories.

In field theories whose configuration space $\{\phi(x)\}$ have a non-trivial topology characterized by a Chern number

$$n = \int dx Q(x) , \tag{3}$$

there exists in fact such an angular variable given by

$$q[\phi] = 2\pi \int_{\phi=0}^{\phi(x)} dx Q(x) , \tag{4}$$

where the integration is to be carried out along a path connecting the two field configurations $\phi = 0$ and $\phi = \phi(x)$, the time variable parametrizing the path *. The angular nature of $q[\phi]$ is a consequence of the fact that, on account of eq. (3), the contribution to eq. (4) of any closed path in configuration space is a multiple of 2π.

* The topological nature of n defined by eq. (3) ensures that the variable $q[\phi]$ is path independent for all paths that can be continuously deformed into each other.

Hence the angular variable defined by eq. (4) parametrizes topologically inequivalent paths in configuration space, and plays the same role as the variable q of our one-dimensional example. The corresponding quantization ambiguity in the field theory case manifests itself through the choice for the configuration space representation of the momentum canonically conjugate to ϕ,

$$\Pi(x) = -i \frac{\delta}{\delta\phi(x)} + \frac{\theta}{2\pi} \frac{\delta q[\phi]}{\delta\phi(x)} . \tag{5}$$

This representation is "gauge" equivalent to the conventional one with multivalued wave functionals satisfying a quasi-periodic boundary condition. This boundary condition characterizes the θ worlds. Expression (5) for the momentum can be derived from a Lagrangian of the form

$$L[\phi, \dot{\phi}] = L_0[\phi, \dot{\phi}] + \frac{\theta}{2\pi} \frac{\mathrm{d}}{\mathrm{d}t} q[\phi] , \tag{6}$$

where the momentum associated with L_0 is represented by $-i\delta/\delta\phi(x)$. The action which corresponds to L is given by

$$S = S_0 + \theta \int \mathrm{d}x Q(x) , \tag{7}$$

which in a Feynman path integral formulation gives rise to the correct θ sectors. θ can be viewed as a measure of the flux through the "hole" in configuration space.

From its definition (4), it is clear that $q[\phi]$ depends on the path history and its range of definition will be the entire real line. In general, however, the same point in configuration space will correspond to different values of $q[\phi]$, and it proves convenient to enlarge configuration space in such a way that different physically equivalent values of the q variable can be associated with different field configurations. This procedure, which corresponds to "unwinding" the configuration space in the field variables will then allow instantons to be interpreted as links between topologically inequivalent classical vacua. In gauge theories this enlargement of configuration space is automatically achieved by working in the temporal gauge, where $q[A]$ can be written in the form

$$q[A] = 2\pi \int \mathrm{d}x I_0(A) . \tag{8}$$

Here I_μ is related to the Chern density by

$$Q = \partial^\mu I_\mu . \tag{9}$$

The path history of the angular variable is now contained in the winding class of the potential A_μ. In contradistinction to the temporal gauge, one finds that in the Coulomb gauge with strong boundary conditions, the variable defined by eq. (8) plays the role of an angular variable whose range is restricted to a 2π interval. In this case a non-trivial path will necessarily involve discontinuities, a mechanism which was exhibited in ref. [4].

From our general discussion on the origin of the θ sectors, it is immediately clear that also in the case of the non-linear O(3) σ model in two dimensions one expects a non-trivial vacuum structure.

In the following discussion we wish to use this model to exemplify the general ideas presented above. The Lagrangian density is given by

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \sigma^a \partial^\mu \sigma^a \,, \tag{10}$$

where $\sigma(x)$ is a three-component unit vector. The topological charge that classifies the field configurations is given by eq. (3) with [7]

$$Q(x) = \frac{1}{8\pi}\, \epsilon^{abc}\epsilon_{\mu\nu}\sigma^a \partial_\mu \sigma^b \partial_\nu \sigma^c \,. \tag{11}$$

Parametrizing the vector σ in terms of a polar angle α and azimuthal angle β, eq. (11) takes the form

$$Q(x) = \partial^\mu I_\mu(x)\,, \tag{12}$$

where

$$I_\mu = \frac{1}{4\pi}\, \epsilon_{\mu\nu} \cos \alpha \partial_\nu \beta \,. \tag{13}$$

It follows that since all points at ∞ are identified in this model, any non-vanishing charge Q must have its origin in singularities of I_μ within the integration domain. Hence the well-known instantons of this model [7] cannot immediately be interpreted as describing tunneling between topologically inequivalent classical vacua if we use the above parametrization. This is already clearly seen by considering the simplest case of a one-instanton configuration centered at $x = 0$ for which α and β are given by *

$$\alpha(x) = 2 \text{ arc cot } \frac{|x-\rho|}{|x+\rho|}\,, \tag{14}$$

$$\beta(x) = \text{arc tan} \frac{2\epsilon_{\mu\nu}\rho_\mu x_\nu}{x^2-\rho^2}\,, \tag{15}$$

where x and ρ are two-component vectors, and the norm of ρ gives the size of the instanton. The singularities of I_μ (eq. (13)) arise from $\partial_\mu\beta$ and are located at the two points $x_\mu = \pm\rho_\mu$. By introducing a cut between these two singularities, $\beta(x)$ may be made to be single valued. In this case $\alpha = \frac{1}{2}\pi$ and $\beta = 0$ at infinity. Across the cut, i.e., precisely in the "strong" field region where the instanton is concentrated, $\beta(x)$ is now discontinuous and jumps by 2π.

This provides one with an explicit example of the discontinuity mechanism discussed in ref. [4] in connection with the Coulomb gauge description of the BPST

* We have chosen the form of the instanton solution given in ref. [8].

instanton, as is most clearly seen by orienting ρ along the space axis. The angular variable corresponding to eq. (8) is given by

$$q[\alpha, \beta] = -\tfrac{1}{2}\int \mathrm{d}x(\partial_x \cos\alpha)\,\beta\,, \qquad (16)$$

and is restricted to a 2π interval. In order to arrive at the analog of the temporal gauge description one must introduce a new degree of freedom which allows one to remove the singularities of I_μ to infinity. To this effect it is natural to employ the formulation of ref. [9] which corresponds to introducing a third unobservable angle γ measuring the rotations about the σ vector. In terms of the angles α, β and γ, the topological charge density may be written in the form

$$Q(x) = \frac{1}{2\pi}\,\epsilon_{\mu\nu}\partial_\mu A_\nu\,, \qquad (17)$$

with

$$A_\mu = \tfrac{1}{2}\cos\alpha\partial_\mu\beta + \partial_\mu\gamma\,. \qquad (18)$$

By choosing γ so that one is in the temporal gauge, the "vector potential" (18) becomes regular at all finite spacetime points and the "unwound" angular variable (4) is given by

$$q[A] = \int \mathrm{d}x A_1\,, \qquad (19)$$

where the path history of q has now been transfered to the winding class of A_1 through

$$A_1(x) = 2\pi \int_{-\infty}^{t} \mathrm{d}t' Q(t', x)\,. \qquad (20)$$

The effective Lagrangian density describing the σ model in the θ sector is given by

$$L_\theta = \tfrac{1}{2}\partial_\mu\sigma^a\partial^\mu\sigma^a + \theta Q\,, \qquad (21)$$

and corresponds to a quantization of this model with a quasi-periodic boundary condition (2) with respect to the angular variable q.

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References

[1] A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, Phys. Lett. 59B (1975) 85.
[2] R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172;
C.G. Callan, Jr., R.F. Dashen and D. Gross, Phys. Lett. 63B (1976) 334.

[3] V.N. Gribov, Materials for the 12th Winter School of the Leningrad Nuclear Research Institute, 1977 (in Russian).
[4] R. Jackiw, I. Muzinich and C. Rebbi, Phys. Rev. D17 (1978) 1576.
[5] M. Ademollo, E. Napolitano and S. Sciuto, Nucl. Phys. B134 (1978) 477.
[6] K.D. Rothe and J.A. Swieca, Nucl. Phys. B138 (1978) 26;
C.W. Bernard and E.J. Weinberg, Phys. Rev. D15 (1978) 3656.
[7] A.A. Belavin and A.M. Polyakov, JETP Lett. 22 (1975) 245.
[8] D.J. Gross, Nucl. Phys. B132 (1978) 439.
[9] A. D'Adda, M. Lüscher and P. Di Vecchia, Nucl. Phys. B146 (1978) 63.